On the dimension of motivic cyclotomic multiple zeta values

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Today's talk

- Introduction and main theorem

- Conjectures

- Sketch of the proof

Introduction and main theorem

Cyclotomic Multiple Zeta Values

Let $\mu_N := \{\zeta_N^a \mid a \in \mathbb{Z}/N\mathbb{Z}\}$ be the set of N-th roots of unity.

Definition (Cyclotomic Multiple Zeta Values (=CMZVs))

For $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$ and $\epsilon_1, \ldots, \epsilon_d \in \mu_N$ such that $(k_d, \epsilon_d) \neq (1,1)$, we define

$$\zeta\begin{pmatrix} k_1, & \ldots, & k_d \\ \epsilon_1, & \ldots, & \epsilon_d \end{pmatrix} := \sum_{0<m_1<\cdots<m_d} \frac{\epsilon_1^{m_1}\cdots\epsilon_d^{m_d}}{m_1^{k_1}\cdots m_d^{k_d}}.$$

- $\zeta\begin{pmatrix} k_1, & \ldots, & k_d \\ 1, & \ldots, & 1 \end{pmatrix} = \sum_{0<m_1<\cdots<m_d} \frac{1}{m_1^{k_1}\cdots m_d^{k_d}} := \zeta(k_1, \ldots, k_d)$

- N : **level.** $k_1 + \cdots + k_d$: **weight.** d : **depth.**

Why do we consider motivic CMZVs ?

Let $Z_w^{(N)} := \langle$ cyclotomic MZVs of level N and weight w $\rangle_{\mathbb{Q}} \subset \mathbb{C}$.

What is $\dim_{\mathbb{Q}} Z_w^{(N)}$?

- In most cases, the determination of $\dim_{\mathbb{Q}} Z_w^{(N)}$ is too difficult because it involves transcendental number theory.

- Thus, we consider the dimension of motivic CMZVs instead for the one of CMZVs.

The motivic CMZVs live in the ring of motivic periods.

$$\begin{array}{ccc} H^{(N)} & \xrightarrow{\rho \text{ (conj. inj.)}} & \mathbb{C} \\ \Cup & & \Cup \\ \zeta^{\mathfrak{m}}\begin{pmatrix} k_1, & \dots, & k_d \\ \epsilon_1, & \dots, & \epsilon_d \end{pmatrix} & \longmapsto & \zeta\begin{pmatrix} k_1, & \dots, & k_d \\ \epsilon_1, & \dots, & \epsilon_d \end{pmatrix} \\ \text{motivic CMZVs} & & \text{CMZVs} \end{array}$$

$H^{(N)} = \bigoplus_{w=0}^{\infty} H_w^{(N)}$: The ring of motivic periods of level N.

(We consider the ring of effective motivic periods of mixed Tate motives associated to $\mathbb{Q}(\mu_N)$ and $\langle 1 - a \mid a \in \mu_N \backslash \{1\} \rangle \otimes \mathbb{Q} \subset \mathbb{Q}(\mu_N)^\times \otimes \mathbb{Q}$. Furthermore, we take the real part when $N = 1,2$.)

Let $Z_w^{(N),\mathfrak{m}} := \langle \text{motivic CMZVs of level } N \text{ and weight } w \rangle_{\mathbb{Q}} \quad \subset H_w^{(N)}$

The dimension of $H_w^{(N)}$

For $N \geq 1$ and $w \geq 0$, define $d_w^{(N)}$ as follows:

$$\sum_{w=0}^{\infty} d_w^{(N)} t^w := \begin{cases} \dfrac{1}{1-t^2-t^3} & N=1 \\ \dfrac{1}{1-t-t^2} & N=2 \\ \dfrac{1}{1-(\frac{\varphi(N)}{2}+\nu(N))t+(\nu(N)-1)t^2} & N \geq 3 \end{cases}$$

(φ is Euler's totient function and $\nu(N)$ is number of prime divisors of N).

Then the dimension of $H_w^{(N)}$ is given by

$$\dim_{\mathbb{Q}} H_w^{(N)} = d_w^{(N)}$$

Upper bound

$Z^{(N),\mathfrak{m}} \subset H_w^{(N)}$ and $\dim_{\mathbb{Q}} H_w^{(N)} = d_w^{(N)}$ implies the following theorem.

Theorem (Deligne-Goncharov)

For $N \geq 1$ and $w \geq 0$, we have $\dim_{\mathbb{Q}} Z_w^{(N),\mathfrak{m}} \leq d_w^{(N)}$.

Consequently, $\dim_{\mathbb{Q}} Z_w^{(N)} \leq d_w^{(N)}$.

The above upper bound is not necessarily strict.

When $\dim_{\mathbb{Q}} Z_w^{(N),\mathfrak{m}} = d_w^{(N)}$ (or equivalently $Z_w^{(N),\mathfrak{m}} = H_w^{(N)}$) ?

Previous results

Theorem (Goncharov 2001)

When $N \geq 5$ is a prime number,

$$\dim_{\mathbb{Q}} Z_2^{(N),\mathfrak{m}} < d_2^{(N)}.$$

Theorem (Deligne 2010 for $N = 2,3,4,8$; Brown 2012 for $N = 1$)

For $N \in \{1,2,3,4,8\}$, we have

$$\dim_{\mathbb{Q}} Z_w^{(N),\mathfrak{m}} = d_w^{(N)} .$$

- The above theorem is equivalent to $Z_w^{(N),\mathfrak{m}} = H_w^{(N)}$ for $N = \{1,2,3,4,8\}$.
- They also give explicit basis for $Z_w^{(N),\mathfrak{m}}$ for $N = \{1,2,3,4,8\}$.

Main theorem and corollaries (1/2)

From now on, let $p \in \{2,3\}$, $q = 6 - p$, $N = qp^r$ $(r \geq 0)$.

Theorem (H. 2024+)

A basis of $H^{(N)}$ is given by

$$\left\{ (2\pi i)^s_{\mathfrak{m}} \zeta^{\mathfrak{m}} \begin{pmatrix} k_1, \ldots, k_d \\ \zeta_N^{a_1}, \ldots, \zeta_N^{a_d} \end{pmatrix} \middle| \begin{array}{r} s, d \geq 0,\ k_1, \ldots, k_d \geq 1,\ a_1, \ldots, a_d \in \mathbb{Z}/N\mathbb{Z} \\ a_1 \equiv \cdots \equiv a_{d-1} \equiv 0,\ a_d \equiv 1 \pmod{q} \end{array} \right\}.$$

Corollary (H.)

We have $\dim_{\mathbb{Q}} Z_w^{(N),\mathfrak{m}} = d_w^{(N)}$ $\quad (= (p^r + 1)^w)$.

Main theorem and corollaries (2/2)

Corollary (H.)

All motivic CMZVs of level N

can be uniquely written as a $\mathbb{Q}$-linear sum of

$$\left\{ (2\pi i)^s_{\mathfrak{m}} \zeta^{\mathfrak{m}} \begin{pmatrix} k_1, \ldots, k_d \\ \zeta_N^{a_1}, \ldots, \zeta_N^{a_d} \end{pmatrix} \middle| \begin{array}{r} s, d \geq 0,\ k_1, \ldots, k_d \geq 1,\ a_1, \ldots, a_d \in \mathbb{Z}/N\mathbb{Z} \\ a_1 \equiv \cdots \equiv a_{d-1} \equiv 0,\ a_d \equiv 1 \pmod{q} \end{array} \right\}.$$

Corollary (H.)

All CMZVs of level N

can be written as a $\mathbb{Q}$-linear sum of

$$\left\{ (2\pi i)^s \zeta \begin{pmatrix} k_1, \ldots, k_d \\ \zeta_N^{a_1}, \ldots, \zeta_N^{a_d} \end{pmatrix} \middle| \begin{array}{r} s, d \geq 0,\ k_1, \ldots, k_d \geq 1,\ a_1, \ldots, a_d \in \mathbb{Z}/N\mathbb{Z} \\ a_1 \equiv \cdots \equiv a_{d-1} \equiv 0,\ a_d \equiv 1 \pmod{q} \end{array} \right\}.$$

Conjectures

Conjectures about the integral structure

Put $\mathbb{Z}_{(p)} = \{a/b \mid b \not\equiv 0 \pmod{p}\}$

Conjecture

For $N = qp^r$ with $p \in \{2,3\}$ and $q = 6 - p$, all CMZVs of level N can be written as a $\mathbb{Z}_{(p)}$-linear sum of

$$\left\{ \frac{(2\pi i/N)^s}{s!} \zeta\begin{pmatrix} k_1, \ldots, k_d \\ \zeta_N^{a_1}, \ldots, \zeta_N^{a_d} \end{pmatrix} \,\middle|\, \begin{array}{r} s, d \geq 0,\ k_1, \ldots, k_d \geq 1,\ a_1, \ldots, a_d \in \mathbb{Z}/N\mathbb{Z} \\ a_1 \equiv \cdots \equiv a_{d-1} \equiv 0,\ a_d \equiv 1 \pmod{q} \end{array} \right\}.$$

Conjecture

All MZVs can be written as a $\mathbb{Z}_{(2)}$-linear sum of

$$\{\zeta(k_1, \ldots, k_d) \mid k_1, \ldots, k_d \in \{2,3\}\}.$$

Other levels

Let $N \geq 3$ be a general and put $D_N := \dim_{\mathbb{Q}}(H_2^{(N)}/Z_2^{(N),\mathfrak{m}})$,

and thus $\dim_{\mathbb{Q}} Z_2^{(N),\mathfrak{m}} = (\varphi(N)/2 + \nu(N) - 1)^2 + \varphi(N) + \nu(N) - D_N$.

- There is an algorithm to calculate D_N.
- Main theorem implies that $D_{2^r} = D_{3^r} = 0$.
- Goncharov shows that $D_p = \dfrac{p^2 - 1}{24}$ for a prime $p \geq 5$.
- It seems those are all known formulas for exact values of D_N.

The case $N = 2p$ or $3p$ for prime p

What are D_{2p} and D_{3p} ?   Is it similar to D_p ?

p	5	7	11	13	17	19	23	29	31	37	41	43	...	83	...	113
D_p	1	2	5	7	12	15	22	35	40	57	70	77	...	330	...	532
D_{2p}	0	0	0	0	1	0	0	0	2	0	1	2	...	0	...	3
D_{3p}	0	0	0	1	0	0	0	0	0	1	4	0	...	0		

Conjecture (H.-Sato)

Let $f(p)$ (resp. $g(p)$) be the order of 2 (resp. 3) in $(\mathbb{Z}/p\mathbb{Z})^\times/\{\pm 1\}$.

Then $D_{2p} = \dfrac{p-1}{2f(p)} - 1$ and $D_{3p} = \dfrac{p-1}{2g(p)} - 1.$

Theorem (H.)

Under the same settings, $D_{2p} \geq \dfrac{p-1}{2f(p)} - 1$ and $D_{3p} \geq \dfrac{p-1}{2g(p)} - 1.$

The case $N = 5p$ for prime p

p	7	11	13	17	19	23	29	31	37	41	43
D_{5p}	0	2	8	8	1	0	16	6	18	23	0

p	47	53	59	61	67	71	73	79	83	89
D_{5p}	46	26	1	94	2	8	36	1	0	317

I have not found any pattern yet.

Other observations

For $N \leq 400$, the following is true.

- $D_{p^2} = \binom{p}{4} = \dfrac{p(p-1)(p-2)(p-3)}{24}$ for a prime p.

  (This was already conjectured by J. Zhao).

- For $a, b \geq 1$ and distinct primes p, q, $\quad D_{p^a q^b} = D_{pq}$.

Sketch of the proof

Hopf structure and coradical filtration (1/2)

Hereafter, we omit (N) from the notation if there is no risk of confusion.

- Put $A := H/(2\pi i)_{\mathfrak{m}}$ and $\zeta^{\mathfrak{a}}(\ldots) = (\zeta^{\mathfrak{m}}(\ldots) \bmod (2\pi i)_{\mathfrak{m}}) \in A$.

- It is known that A has a Hopf algebra structure:

  $\Delta : A \otimes A \to A$.

- Then Δ define the coradical filtration

$$\{0\} = C_{-1}A \subset C_0A \subset C_1A \subset C_2A \subset \cdots \subset A$$

- Put $\mathrm{gr}^C_d A := C_dA / C_{d-1}A$.

- Δ induces an isomorphism $\mathrm{gr}^C_d A \simeq \mathrm{gr}^C_{d-1} A \otimes \mathrm{gr}^C_1 A \simeq \cdots \simeq (\mathrm{gr}^C_1 A)^{d\otimes}$.

Hopf structure and coradical filtration (2/2)

- We write $\zeta^C\begin{pmatrix} k_1, & \ldots, & k_d \\ \epsilon_1, & \ldots, & \epsilon_d \end{pmatrix}$ for the image of $\zeta^{\mathfrak{a}}\begin{pmatrix} k_1, & \ldots, & k_d \\ \epsilon_1, & \ldots, & \epsilon_d \end{pmatrix}$ in $\mathrm{gr}_d^C A$.

- The following is a refinement of the main theorem.

Theorem (H.)

For $w \geq d \geq 1$, a basis of $\mathrm{gr}_d^C A_w$ is given by

$$\left\{ \zeta^C\begin{pmatrix} k_1, \ldots, k_d \\ \zeta_N^{a_1}, \ldots, \zeta_N^{a_d} \end{pmatrix} \middle| \begin{array}{l} k_1 + \cdots + k_d = w,\ a_1, \ldots, a_d \in \mathbb{Z}/N\mathbb{Z} \\ a_1 \equiv \cdots \equiv a_{d-1} \equiv 0,\ a_d \equiv 1 \pmod{q} \end{array} \right\}.$$

The proof of main theorem is carried out by a purely combinatorial argument.

We only explain the case $w = d$ for simplicity.

Note that $\mathrm{gr}_d^C A_d \simeq A_1^{\otimes d}$ in this case.

Structure of A_1

For $a \in \mathbb{Z}/N\mathbb{Z}$, put $\langle a \rangle := \zeta^{\mathfrak{a}}\begin{pmatrix} 1 \\ \zeta_N^a \end{pmatrix}$.

Theorem (Deligne-Goncharov)

$A_1^{(N)}$ is spanned by $\langle a \rangle$ for $a \in \mathbb{Z}/N\mathbb{Z}$ with the following relations:

- $\langle 0 \rangle = 0,$
- $\langle a \rangle = \langle -a \rangle$ for $a \in \mathbb{Z}/N\mathbb{Z}$,
- $\langle b \rangle = \sum_{aM=b} \langle a \rangle$ for $M \mid N$ and $b \neq 0 \in M\mathbb{Z}/N\mathbb{Z}$.

- Let $N = qp^r$ with $p \in \{2,3\}$, $q = 6 - p$.

- Put $W := \bigoplus_{\substack{a \in \mathbb{Z}/N\mathbb{Z} \\ a \equiv 1 \pmod q}} \mathbb{Q}[a] \subset \mathbb{Q}[\mathbb{Z}/N\mathbb{Z}]$.

- We can show that the map $W \to A_1; [a] \mapsto \langle a \rangle$ is bijective.

  We denote by $\theta : A_1 \to W$ the inverse of this bijection.

- We write $[a_1, \dots, a_e]$ for $[a_1] \otimes \cdots \otimes [a_d]$.

- Define $\rho = \rho_d : W^{\otimes d} \to W^{\otimes d}$ by

$$\rho([a_1, \dots, a_d]) = [a_1, \dots, a_d] + \sum_{i=2}^{d} [a_1, \dots, \widehat{a_i}, \dots, a_d] \otimes \theta(\langle a_{i-1} - a_i \rangle) - \sum_{i=1}^{d-1} [a_1, \dots, \widehat{a_i}, \dots, a_d] \otimes \theta(\langle a_{i+1} - a_i \rangle)$$

Proof of main theorem (1/4)

Let $\bar{\rho}$ be a modulo p reduction of ρ.

Main theorem (for $w = d$ case)

$\Leftrightarrow$ ρ is bijective

$\Leftrightarrow$ $\det(\rho) \neq 0$

$\Leftarrow$ $\det(\rho) \not\equiv 0 \pmod{p}$

$\Leftrightarrow$ $\bar{\rho}$ is bijective

$\Leftarrow$ $\bar{\rho}$ is unipotent

Thus main theorem is proved if the unipotency of $\bar{\rho}$ is proved.

(When $N \in \{3,4,8\}$, $\bar{\rho} = \mathrm{id}$)

Proof of main theorem (2/4)

- Let $W_p := \bigoplus_{\substack{a \in \mathbb{Z}/N\mathbb{Z} \\ a \equiv 1 \pmod q}} \mathbb{F}_p[a] \subset \mathbb{F}_p[\mathbb{Z}/N\mathbb{Z}]$ be a mod p reduction of W.

- Define a subgroup G of $(\mathbb{Z}/N\mathbb{Z})^\times$ by $G = \{\sigma \equiv 1 \pmod q\}$

- For $\sigma_1, \ldots, \sigma_d \in G$, define $\Phi_{\sigma_1,\ldots,\sigma_d} : W_p^{\otimes d} \to W_p^{\otimes d}$ by

  $\Phi_{\sigma_1,\ldots,\sigma_d}([a_1, \ldots, a_d]) := [\tau_1 a_1, \ldots, \tau_d a_d]$ where $\tau_j = \prod_{i=1}^{j} \sigma_i$.

- By linearity, we extend the definition above to $\Phi_{g_1,\ldots,g_d}$ for $g_1, \ldots, g_d \in \mathbb{F}_p[G]$.

Proof of main theorem (3/4)

- For $J = (m_1, n_1, \ldots, m_d, n_d) \in \mathbb{Z}^{2d}$, define $V_J \in W_p^{\otimes d}$ as the subspace spanned by elements of the forms

$$\Phi_{g_1,\ldots,g_d}\left(\sum_{\eta_i \in U(n_i)} [a_1 + \eta_1, \ldots, a_d + \eta_d]\right) \text{ where}$$

  - $U(n) := \begin{cases} (p^{-n}N)\mathbb{Z}/N\mathbb{Z} & n \leq r \\ \varnothing & n > r \end{cases}$
  - $g_i \in I_G^{m_i}$ where $I_G := \ker(\mathbb{F}_p[G] \to \mathbb{F}_p)$.
  - $a_1, \ldots, a_d \equiv 1 \mod q$

Note that $V_J = \{0\}$ for all but finitely many J.

Proof of main theorem (4/4)

We say that $J = (n_1, m_1, \ldots, n_d, m_d) \in \mathbb{Z}^{2d}$ is proper if and only if:

- $n_i \le n_{i+1}$ for $i = 1, \ldots, d-1$.
- $n_i = n_{i+1} \Rightarrow m_i = 0$ for $i = 1, \ldots, d-1$.

Lemma

For a proper $J = (m_1, n_1, \ldots, m_d, n_d)$ and $x \in V_J$, we have

$$(\bar{\rho} - \mathrm{id})(v) \in \sum_{J'} V_{J'}$$

where J' runs over all proper tuples greater than J in the lexicographic order.

This lemma implies the unipotency of $\bar{\rho}$, which completes the proof!

Thank you for listening!

谢谢！

Appendix - Table of D_N for $N \leq 250$

N	3	4	5	6	7	8	9	10	11	12	13	14
D_N	0	0	1	0	2	0	0	0	5	0	7	0

N	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
D_N	0	0	12	0	15	0	0	0	22	0	5	0	0	0	35

N	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
D_N	0	40	0	0	1	0	0	57	0	1	0	70	0	77	0

N	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
D_N	0	0	92	0	35	0	0	0	117	0	2	0	0	0	145

N	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
D_N	0	155	2	0	0	8	0	187	1	0	0	210	0	222	0

N	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89
D_N	0	0	15	0	260	0	0	1	287	0	8	2	0	0	330

Appendix - Table of D_N for $N \leq 250$

N	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104
D_N	0	32	0	0	0	1	0	392	0	0	0	425	0	442	0

N	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119
D_N	0	0	477	0	495	0	1	0	532	0	0	0	1	0	72

N	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134
D_N	0	330	0	4	2	25	0	672	0	0	0	715	0	99	0

N	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149
D_N	0	1	782	0	805	0	0	0	240	0	16	3	0	0	925

N	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164
D_N	0	950	0	0	0	6	0	1027	0	0	0	165	0	1107	1

N	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179
D_N	0	0	1162	0	715	0	0	2	1247	0	0	0	0	3	1335

Appendix - Table of D_N for $N \leq 250$

N	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194
D_N	0	1365	0	5	0	18	0	440	0	0	0	1520	0	1552	1

N	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209
D_N	0	0	1617	0	1650	0	2	0	294	0	23	0	0	0	585

N	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224
D_N	0	1855	0	0	0	0	0	345	2	5	0	720	0	2072	0

N	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239
D_N	0	3	2147	0	2185	0	0	0	2262	0	46	0	0	0	2380

N	240	241	242	243	244	245	246	247	248	249	250
D_N	0	2420	0	0	0	0	0	954	2	0	0

See arXiv:2408.15975 for further values.